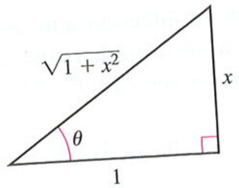


$\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$
$\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$

FIGURE 7.9

The final step is to return to the original variable x. The first term $\theta/2$ is replaced by $\frac{1}{2}\tan^{-1} x$. The second term involving $\sin 2\theta$ requires the identity $\sin 2\theta = 2\sin\theta\cos\theta$. The reference triangle (Figure 7.9) tells us that

$$\frac{1}{4}\sin 2\theta = \frac{1}{2}\sin\theta\cos\theta = \frac{1}{2}\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}} = \frac{1}{2}\cdot\frac{x}{1+x^2}.$$

The integration can now be completed:

$$\int\frac{dx}{(1+x^2)^2} = \left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)+C$$
$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C$$

Related Exercises 15–46 ◂

EXAMPLE 5 A secant substitution Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$.

SOLUTION This example illustrates a useful preliminary step before making a trigonometric substitution. The integrand does not contain any of the patterns in Table 7.3 that suggest a trigonometric substitution. Completing the square does, however, lead to one of those patterns. Noting that $x^2+4x-5=(x+2)^2-9$, we change variables with $u = x+2$ and write the integral as

$$\int_1^4\frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4\frac{\sqrt{(x+2)^2-9}}{x+2}\,dx \qquad \text{Complete the square.}$$
$$= \int_3^6\frac{\sqrt{u^2-9}}{u}\,du \qquad u = x+2,\ du = dx \quad \text{Change limits of integration.}$$

➤ Recall that to complete the square with x^2+bx+c, you add and subtract $(b/2)^2$ to the expression, and then factor to form a perfect square. You could also make the single substitution $x+2 = 3\sec\theta$ in Example 5.

This new integral calls for the secant substitution $u = 3\sec\theta$ (where $0 \le \theta < \pi/2$), which implies that $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$. We also change the limits of integration: When $u=3$, $\theta=0$, and when $u=6$, $\theta=\pi/3$. The complete integration can now be done:

➤ The substitution $u = 3\sec\theta$ can be rewritten as $\theta = \sec^{-1}(u/3)$. Because $u \ge 3$ in the integral $\displaystyle\int_3^6\frac{\sqrt{u^2-9}}{u}\,du$, we have $0 \le \theta < \frac{\pi}{2}$.

$$\int_1^4\frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_3^6\frac{\sqrt{u^2-9}}{u}\,du \qquad u = x+2,\ du = dx$$
$$= \int_0^{\pi/3}\frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \qquad u = 3\sec\theta,\ du = 3\sec\theta\tan\theta\,d\theta$$
$$= 3\int_0^{\pi/3}\tan^2\theta\,d\theta \qquad \text{Simplify.}$$
$$= 3\int_0^{\pi/3}(\sec^2\theta-1)\,d\theta \qquad \tan^2\theta = \sec^2\theta - 1$$
$$= 3(\tan\theta-\theta)\Big|_0^{\pi/3} \qquad \text{Evaluate integrals.}$$
$$= 3\sqrt{3}-\pi \qquad \text{Simplify.}$$

Related Exercises 15–46 ◂

SECTION 7.3 EXERCISES

Review Questions

1. What change of variables is suggested by an integral containing $\sqrt{x^2-9}$?
2. What change of variables is suggested by an integral containing $\sqrt{x^2+36}$?
3. What change of variables is suggested by an integral containing $\sqrt{100-x^2}$?
4. If $x = 4\tan\theta$, express $\sin\theta$ in terms of x.
5. If $x = 2\sin\theta$, express $\cot\theta$ in terms of x.
6. If $x = 8\sec\theta$, express $\tan\theta$ in terms of x.

Basic Skills

7–10. *Evaluate the following integrals.*

7. $\displaystyle\int_0^{5/2}\frac{dx}{\sqrt{25-x^2}}$
8. $\displaystyle\int_0^{3/2}\frac{dx}{(9-x^2)^{3/2}}$
9. $\displaystyle\int_5^{10}\sqrt{100-x^2}\,dx$
10. $\displaystyle\int_0^{\sqrt{2}}\frac{x^2}{\sqrt{4-x^2}}\,dx$

11–14. *Evaluate the following integrals.*

11. $\displaystyle\int\frac{dx}{(16-x^2)^{1/2}}$
12. $\displaystyle\int\sqrt{36-x^2}\,dx$
13. $\displaystyle\int\frac{\sqrt{9-x^2}}{x}\,dx$
14. $\displaystyle\int(36-9x^2)^{-3/2}\,dx$

15–40. *Evaluate the following integrals.*

15. $\displaystyle\int\sqrt{64-x^2}\,dx$
16. $\displaystyle\int\frac{dx}{\sqrt{x^2-49}},\ x>7$
17. $\displaystyle\int\frac{dx}{\sqrt{36-x^2}}$
18. $\displaystyle\int\frac{dx}{\sqrt{16+4x^2}}$
19. $\displaystyle\int\frac{dx}{\sqrt{x^2-81}},\ x>9$
20. $\displaystyle\int\frac{dx}{\sqrt{1-2x^2}}$
21. $\displaystyle\int\frac{dx}{(1+4x^2)^{3/2}}$
22. $\displaystyle\int\frac{dx}{(x^2-36)^{3/2}},\ x>6$
23. $\displaystyle\int\frac{x^2}{\sqrt{16-x^2}}\,dx$
24. $\displaystyle\int\frac{dx}{(81+x^2)^2}$
25. $\displaystyle\int\frac{\sqrt{x^2-9}}{x}\,dx,\ x>3$
26. $\displaystyle\int\sqrt{9-4x^2}\,dx$
27. $\displaystyle\int\frac{x^2}{\sqrt{4+x^2}}\,dx$
28. $\displaystyle\int\frac{\sqrt{4x^2-1}}{x^2}\,dx,\ x>\frac{1}{2}$
29. $\displaystyle\int\frac{dx}{\sqrt{3-2x-x^2}}$
30. $\displaystyle\int\frac{x^4}{1+x^2}\,dx$
31. $\displaystyle\int\frac{\sqrt{9x^2-25}}{x^3}\,dx,\ x>\frac{5}{3}$
32. $\displaystyle\int\frac{\sqrt{9-x^2}}{x^2}\,dx$
33. $\displaystyle\int\frac{x^2}{(25+x^2)^2}\,dx$
34. $\displaystyle\int\frac{dx}{x^2\sqrt{9x^2-1}},\ x>\frac{1}{3}$
35. $\displaystyle\int\frac{x^2}{(100-x^2)^{3/2}}\,dx$
36. $\displaystyle\int\frac{dx}{x^3\sqrt{x^2-100}},\ x>10$
37. $\displaystyle\int\frac{x^3}{(81-x^2)^2}\,dx$
38. $\displaystyle\int\frac{dx}{x^3\sqrt{x^2-1}},\ x>1$
39. $\displaystyle\int\frac{dx}{x(x^2-1)^{3/2}},\ x>1$
40. $\displaystyle\int\frac{x^3}{(x^2-16)^{3/2}}\,dx,\ x<-4$

41–46. Evaluating definite integrals *Evaluate the following definite integrals.*

41. $\displaystyle\int_0^1\frac{dx}{\sqrt{x^2+16}}$
42. $\displaystyle\int_{8\sqrt{2}}^{16}\frac{dx}{\sqrt{x^2-64}}$
43. $\displaystyle\int_0^{1/3}\frac{dx}{(9x^2+1)^{3/2}}$
44. $\displaystyle\int_{10/\sqrt{3}}^{10}\frac{dx}{\sqrt{x^2-25}}$
45. $\displaystyle\int_{4/\sqrt{3}}^{4}\frac{dx}{x^2(x^2-4)}$
46. $\displaystyle\int_6^{6\sqrt{3}}\frac{x^2}{(x^2+36)^2}\,dx$

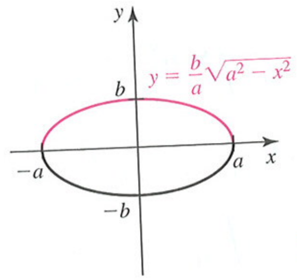
Further Explorations

47. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    a. If $x = 4\tan\theta$, then $\csc\theta = 4/x$.
    b. The integral $\int_1^2\sqrt{1-x^2}\,dx$ does not have a finite real value.
    c. The integral $\int_1^2\sqrt{x^2-1}\,dx$ does not have a finite real value.
    d. The integral $\displaystyle\int\frac{dx}{x^2+4x+9}$ cannot be evaluated using a trigonometric substitution.

48–55. Completing the square *Evaluate the following integrals.*

48. $\displaystyle\int\frac{dx}{x^2-2x+10}$
49. $\displaystyle\int\frac{dx}{x^2+6x+18}$
50. $\displaystyle\int\frac{dx}{2x^2-12x+36}$
51. $\displaystyle\int\frac{x^2-2x+1}{\sqrt{x^2-2x+10}}\,dx$
52. $\displaystyle\int\frac{x^2+2x+4}{\sqrt{x^2-4x}}\,dx,\ x>4$
53. $\displaystyle\int\frac{x^2-8x+16}{(9+8x-x^2)^{3/2}}\,dx$
54. $\displaystyle\int_1^4\frac{dx}{x^2-2x+10}$
55. $\displaystyle\int_{1/2}^{(\sqrt{2}+3)/(2\sqrt{2})}\frac{dx}{8x^2-8x+11}$

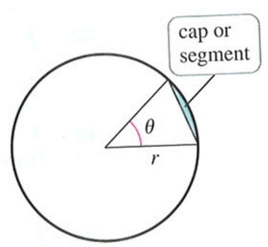
56. Area of an ellipse The upper half of the ellipse centered at the origin with axes of length $2a$ and $2b$ is described by $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (see figure). Find the area of the ellipse in terms of a and b.



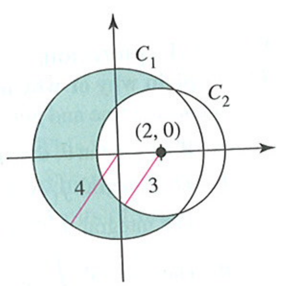
57. Area of a segment of a circle Use two approaches to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ (see figure) is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta).$$

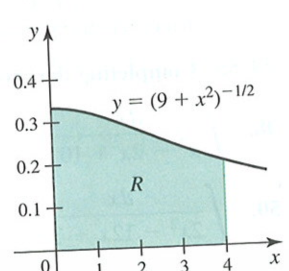
a. Find the area using geometry (no calculus).
b. Find the area using calculus.



58. Area of a lune A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2, 0)$. Find the area of the lune (shaded in the figure) that lies inside C_1 and outside C_2.



59. Area and volume Consider the function $f(x) = (9 + x^2)^{-1/2}$ and the region R on the interval $[0, 4]$ (see figure).
a. Find the area of R.
b. Find the volume of the solid generated when R is revolved about the x-axis.
c. Find the volume of the solid generated when R is revolved about the y-axis.



T **60. Area of a region** Graph the function $f(x) = (16 + x^2)^{-3/2}$ and find the area of the region bounded by the curve and the x-axis on the interval $[0, 3]$.

61. Arc length of a parabola Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

62. Comparing areas On the interval $[0, 2]$, the graphs of $f(x) = x^2/3$ and $g(x) = x^2(9 - x^2)^{-1/2}$ have similar shapes.
a. Find the area of the region bounded by the graph of f and the x-axis on the interval $[0, 2]$.
b. Find the area of the region bounded by the graph of g and the x-axis on the interval $[0, 2]$.
c. Which region has the greater area?

T **63–65. Using the integral of $\sec^3 u$** *By reduction formula 4 in Section 7.2,*

$$\int \sec^3 u\, du = \frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + C.$$

Graph the following functions and find the area under the curve on the given interval.

63. $f(x) = (9 - x^2)^{-2}$, $\left[0, \frac{3}{2}\right]$ **64.** $f(x) = (4 + x^2)^{1/2}$, $[0, 2]$

65. $f(x) = (x^2 - 25)^{1/2}$, $[5, 10]$

66–67. Asymmetric integrands *Evaluate the following integrals. Consider completing the square.*

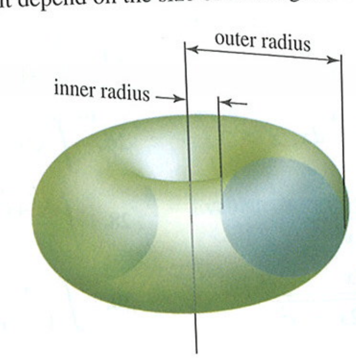
66. $\displaystyle\int \frac{dx}{\sqrt{(x-1)(3-x)}}$ **67.** $\displaystyle\int_{2+\sqrt{2}}^{4} \frac{dx}{\sqrt{(x-1)(x-3)}}$

68. Clever substitution Evaluate $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$ using the substitution $x = 2\tan^{-1}\theta$. The identities $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ are helpful.

Applications

69. A torus (doughnut) Find the volume of the solid torus formed when the circle of radius 4 centered at $(0, 6)$ is revolved about the x-axis.

70. Bagel wars Bob and Bruce bake bagels (shaped like tori). They both make standard bagels that have an inner radius of 0.5 in and an outer radius of 2.5 in. Bob plans to increase the volume of his bagels by decreasing the inner radius by 20% (leaving the outer radius unchanged). Bruce plans to increase the volume of his bagels by increasing the outer radius by 20% (leaving the inner radius unchanged). Whose new bagels will have the greater volume? Does this result depend on the size of the original bagels? Explain.



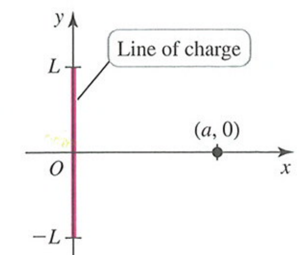
71. Electric field due to a line of charge A total charge of Q is distributed uniformly on a line segment of length $2L$ along the y-axis (see figure). The x-component of the electric field at a point $(a, 0)$ on the x-axis is given by

$$E_x(a) = \frac{kQa}{2L}\int_{-L}^{L} \frac{dy}{(a^2 + y^2)^{3/2}},$$

where k is a physical constant and $a > 0$.

a. Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$.
b. Letting $\rho = Q/2L$ be the charge density on the line segment, show that if $L \to \infty$, then $E_x(a) = 2k\rho/a$.

(See the Guided Projects for a derivation of this and other similar integrals.)

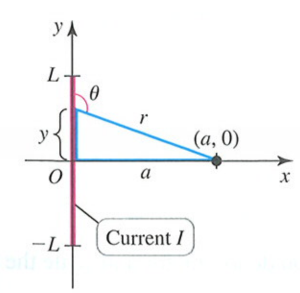


72. Magnetic field due to current in a straight wire A long straight wire of length $2L$ on the y-axis carries a current I. According to the Biot-Savart Law, the magnitude of the magnetic field due to the current at a point $(a, 0)$ is given by

$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}\, dy$$

where μ_0 is a physical constant, $a > 0$, and θ, r, and y are related as shown in the figure.

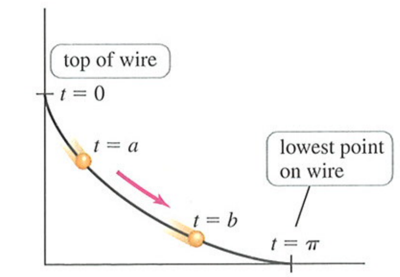
a. Show that the magnitude of the magnetic field at $(a, 0)$ is $B(a) = \dfrac{\mu_0 I L}{2\pi a\sqrt{a^2 + L^2}}$.
b. What is the magnitude of the magnetic field at $(a, 0)$ due to an infinitely long wire $(L \to \infty)$?



73. Fastest descent time The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid (see figure). A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time (see the Guided Project The Amazing Cycloid for more about this *brachistochrone property*). It can be shown that the descent time between any two points $0 \le a \le b \le \pi$ on the curve is

$$\text{descent time} = \int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}}\, dt,$$

where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.



a. Find the descent time on the interval $[a, b]$ by making the substitution $u = \cos t$.
b. Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

T **74. Maximum path length of a projectile** (Adapted from Putnam Exam 1940) A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the x-axis is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y = -\frac{1}{2}kx^2 + y_{max}, \quad \text{where } k = \frac{g}{(V\cos\theta)^2}$$

$$\text{and } y_{max} = \frac{(V\sin\theta)^2}{2g}.$$

a. Note that the high point of the trajectory occurs at $(0, y_{max})$. If the projectile is on the ground at $(-a, 0)$ and $(a, 0)$, what is a?
b. Show that the length of the trajectory (arc length) is $2\int_0^a \sqrt{1 + k^2x^2}\, dx$.
c. Evaluate the arc length integral and express your result in terms of V, g, and θ.
d. For a fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta + \tan\theta) = 1$.
e. Use a graphing utility to approximate the optimal launch angle.

Additional Exercises

T **75–78. Care with the secant substitution** *Recall that the substitution $x = a\sec\theta$ implies that $x \ge a$ (in which case $0 \le \theta < \pi/2$ and $\tan\theta \ge 0$) or $x \le -a$ (in which case $\pi/2 < \theta \le \pi$ and $\tan\theta \le 0$).*

75. Show that

$$\int \frac{dx}{x\sqrt{x^2-1}} = \begin{cases} \sec^{-1} x + C = \tan^{-1}\sqrt{x^2-1} + C & \text{if } x > 1 \\ -\sec^{-1} x + C = -\tan^{-1}\sqrt{x^2-1} + C & \text{if } x < -1 \end{cases}$$

76. Evaluate for $\int \frac{\sqrt{x^2-1}}{x^3}\,dx$ for $x > 1$ and for $x < -1$.

T 77. Graph the function $f(x) = \frac{\sqrt{x^2-9}}{x}$ and consider the region bounded by the curve and the x-axis on $[-6, -3]$. Then, evaluate $\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}\,dx$. Be sure the result is consistent with the graph.

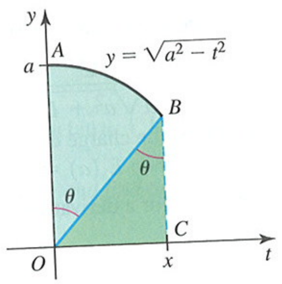
T 78. Graph the function $f(x) = \frac{1}{x\sqrt{x^2-36}}$ on its domain. Then, find the area of the region R_1 bounded by the curve and the x-axis on $[-12, -12/\sqrt{3}]$ and the region R_2 bounded by the curve and the x-axis on $[12/\sqrt{3}, 12]$. Be sure your results are consistent with the graph.

79. Visual Proof Let $F(x) = \int_0^x \sqrt{a^2 - t^2}\,dt$. The figure shows that $F(x) =$ area of sector OAB + area of triangle OBC.

a. Use the figure to prove that $F(x) = \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2}.$

b. Conclude that $\int \sqrt{a^2 - x^2}\,dx = \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2} + C.$

[*Source: The College Mathematics Journal* 34, no. 3 (May 2003)]



QUICK CHECK ANSWERS

1. Use $x = 3\sin\theta$ to obtain $9\cos^2\theta$. **2.** (a) Use $x = 3\tan\theta$. (b) Use $x = 4\sin\theta$. **3.** Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\,d\theta$. The new integral is $\int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$ ◀

7.4 Partial Fractions

Later in this chapter, we will see that finding the velocity of a skydiver requires evaluating an integral of the form $\int \frac{dv}{a - bv^2}$, where a and b are constants. Similarly, finding the population of a species that is limited in size involves an integral of the form $\int \frac{dP}{aP(1-bP)}$, where a and b are constants. These integrals have the common feature that their integrands are rational functions. Similar integrals result from modeling mechanical and electrical networks. The goal of this section is to introduce the *method of partial fractions* for integrating rational functions. When combined with standard and trigonometric substitutions, this method allows us (in principle) to integrate any rational function.

➤ Recall that a rational function has the form p/q, where p and q are polynomials.

Method of Partial Fractions

Given a function such as

$$f(x) = \frac{1}{x-2} + \frac{2}{x+4},$$

it is a straightforward task to find a common denominator and write the equivalent expression

$$f(x) = \frac{(x+4) + 2(x-2)}{(x-2)(x+4)} = \frac{3x}{(x-2)(x+4)} = \frac{3x}{x^2+2x-8}.$$

The purpose of partial fractions is to reverse this process. Given a rational function that is difficult to integrate, the method of partial fractions produces an equivalent function that is much easier to integrate.

rational function	*method of partial fractions*	*partial fraction decomposition*
$\frac{3x}{(x-2)(x+4)} = \frac{3x}{x^2+2x-8}$	→	$\frac{1}{x-2} + \frac{2}{x+4}$
Difficult to integrate: $\int \frac{3x}{x^2+2x-8}\,dx$		**Easy to integrate:** $\int \left(\frac{1}{x-2} + \frac{2}{x+4}\right) dx$

QUICK CHECK 1 Find an antiderivative of $f(x) = \frac{1}{x-2} + \frac{2}{x+4}$. ◀

The Key Idea Working with the same function, $f(x) = \frac{3x}{(x-2)(x+4)}$, our objective is to write it in the form

$$\frac{A}{x-2} + \frac{B}{x+4},$$

where A and B are constants to be determined. This expression is called the **partial fraction decomposition** of the original function; in this case, it has two terms, one for each factor in the denominator of the original function.

➤ Notice that the numerator of the original rational function does not affect the form of the partial fraction decomposition. The constants A and B are called *undetermined coefficients*.

The constants A and B are determined using the condition that the original function f and its partial fraction decomposition must be equal for all values of x; that is,

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}. \quad (1)$$

Multiplying both sides of equation (1) by $(x-2)(x+4)$ gives

$$3x = A(x+4) + B(x-2).$$

➤ This step requires that $x \neq 2$ and $x \neq -4$; both values are outside the domain of f.

Collecting like powers of x results in

$$3x = (A+B)x + (4A - 2B). \quad (2)$$

If equation (2) is to hold for all values of x, then

- the coefficients of x^1 on both sides of the equation must match;
- the coefficients of x^0 (that is, the constants) on both sides of the equation must match.

$$3x + 0 = (A+B)x + (4A - 2B)$$

This observation leads to two equations for A and B.

Match coefficients of x^1: $3 = A + B$
Match coefficients of x^0: $0 = 4A - 2B$

The first equation says that $A = 3 - B$. Substituting $A = 3 - B$ into the second equation gives the equation $0 = 4(3 - B) - 2B$. Solving for B, we find that $6B = 12$, or $B = 2$. The value of A now follows; we have $A = 3 - B = 1$.

Substituting these values of A and B into equation (1), the partial fraction decomposition is

$$\frac{3x}{(x-2)(x+4)} = \frac{1}{x-2} + \frac{2}{x+4}.$$